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Field Theory of Critical Behavior in a Driven Diffusive System

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We use a field theoretic renormalization group method to study the critical properties of a diffusive system with a single conserved density subject to a constant uniform external field. A fixed point stable below $d_c = 5$ is found to govern the critical behavior. Scaling forms of density correlation functions are derived and critical exponents are obtained to all orders in $\varepsilon = 5 - d$. Spatial correlations are found to be very anisotropic with elongated correlations along the external field. Long wavelength transverse fluctuations are suppressed completely to yield mean field transverse exponents.

KEY WORDS: Nonequilibrium steady states; driven diffusive systems; phase transitions; field theoretic renormalization group; fluctuation-dissipation theorem.

1. INTRODUCTION

Systems in nonequilibrium steady states subject to an external driving force have attracted some recent interest.[1-5] Results of computer simulations[1] of stochastic lattice-gas models in two and three dimensions suggest that there exists a phase transition from a disordered phase to an ordered one, where the ordering is very anisotropic with striplike configurations parallel to the external field. A theoretical study[2] of a slightly different version of the discrete model in the limit of infinite ratio of jump rates shows phase transitions of mean-field character.

Recently van Beijeren *et al.*,[3] using a mode-coupling approximation, and Janssen and Schmittmann,[4] using field theoretic techniques, considered the continuum version of these driven diffusive systems. They studied the long-time behavior of density fluctuations above the critical

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point. Anomalous diffusive behavior was found for $d \leqslant 2$. Also Gawedzki and Kupiainen[5] studied these dynamical systems under external fields of different symmetries using a supersymmetric formulation. Their results show a crossover to critical behavior with upper critical dimension $d_c = 5$ (or 3 for different symmetry of the external field). However, these authors have not analyzed the critical properties, which is the main purpose of the present work.

We formulate the problem in a field theoretic renormalization group (RG) approach as in Janssen and Schmittmann. Only the longitudinal diffusion coefficient is found to get renormalized by the fluctuations. This coefficient will then vanish at a lower temperature than that of the transverse diffusion coefficient. The system consequently exhibits critical behavior governed by a fixed point with $d_c = 5$. Very surprisingly most of the ultraviolet divergences are absent there, enabling us to determine critical exponents to all orders in $\varepsilon = 5 - d$. The results suggest strong spatial anisotropy, with elongated correlations along the direction of the driving force.

The exponents describing the transverse correlations are found to be mean-field valued, reflecting the suppression and distortion of the long wavelength transverse fluctuations by the driving force, similar to the case of fluids under shear flow.[6] The form of the scaling relations between exponents is modified by extra terms which vanish in the absence of anisotropy. It is also necessary to introduce a new exponent to express the possible breakdown of the fluctuation-dissipation theorem (FDT).[7] However, this exponent turns out to be zero at the above-mentioned fixed point, indicating that the FDT is satisfied by the most singular parts of the functions involved.

We also obtain another fixed point with $d_c = 4.5$ in a different region of the parameter space. For completeness we also list the corresponding results to order one-loop.

This paper is organized as follows: In Sec. 2 we outline the dynamical formulation of the field theory of driven diffusive systems with a single conserved density. We present in Sec. 3 the one-loop calculations of various renormalization constants in different regions of the parameter space. Section 4 then discusses the scaling forms of the density correlation functions and defines a complete set of critical exponents to describe the critical behavior. Scaling relations and exponents are presented. We summarize and make some remarks on detailed balance symmetry in Sec. 5. Finally the appendices contain evaluations of one-loop integrals and a calculation of the crossover exponent from the $E = 0$ fixed point to the $E \neq 0$ fixed point. Also the absence of certain ultraviolet divergences is discussed and a calculation of the exponent β is presented.

2. FIELD THEORETIC FORMULATION

Let us consider a diffusive system under a constant uniform external driving force which maintains the system in a nonequilibrium steady state. In a continuum model, the particle density $c(\mathbf{x}, t)$ at position $\mathbf{x}$ and time t obeys the equation of motion[3]

$$\frac{\partial}{\partial t} c(\mathbf{x}, t) + \nabla \cdot \mathbf{j}(\mathbf{x}, t) = 0 \tag{2.1a}$$

$$\mathbf{j}(\mathbf{x}, t) = -D(c(\mathbf{x}, t)) \nabla c(\mathbf{x}, t) + c(\mathbf{x}, t)\, \mathbf{u}(c(\mathbf{x}, t)) + \mathbf{j}_L(\mathbf{x}, t) \tag{2.1b}$$

where $\mathbf{j}(\mathbf{x}, t)$ is the particle current, $D(c)$ the diffusion coefficient, $\mathbf{u}(c)$ the velocity field which accounts for the action of the driving force, and $\mathbf{j}_L$ a Gaussian white noise contribution which is supposed to summarize the effects of microscopic degrees of freedom.

In the language of Ising magnets, the magnetization $\varphi(\mathbf{x}, t)$ corresponds to $c(\mathbf{x}, t) - \bar{c}$, where $\bar{c}$ is the uniform average particle density. In the absence of an external magnetic field, $\varphi(\mathbf{x}, t)$ satisfies the kinetic equation[4]

$$\begin{aligned}\frac{\partial}{\partial t} \varphi(\mathbf{x}, t) = {} & \frac{1}{2} \lambda E\, \partial_\parallel \varphi^2(\mathbf{x}, t) \\ & + \lambda[r_\parallel \partial_\parallel^2 + r_\perp \nabla_\perp^2 - \rho\, \partial_\parallel^4 - \partial_\parallel^2 \nabla_\perp^2 \\ & - \eta(\nabla_\perp^2)^2]\, \varphi(\mathbf{x}, t) + \zeta(\mathbf{x}, t)\end{aligned} \tag{2.2a}$$

$$\langle \zeta(\mathbf{x}, t)\, \zeta(\mathbf{x}', t') \rangle = -2\lambda(\sigma \partial_\parallel^2 + \nabla_\perp^2)\, \delta(\mathbf{x} - \mathbf{x}')\, \delta(t - t') \tag{2.2b}$$

This is obtained from (2.1) (cf. Refs. 3 and 4): The first term comes from expanding $c\mathbf{u}(c)$ to order φ^2 and performing a suitable Galilean transformation, the second term is the diffusion current, and $\zeta(\mathbf{x}, t) = -\nabla \cdot \mathbf{j}_L(\mathbf{x}, t)$. λ is the transport coefficient; $r_\parallel$ and $r_\perp$ are the reduced temperature variables of the form $T - T_\parallel$ and $T - T_\perp$ respectively; and ρ, η, and σ are parameters taken to be independent of temperature. They take care of anisotropies which the external field E induces in the diffusion coefficient and the noise correlations.

Equation (2.2) is the most general one for this system, being consistent with the following symmetries associated with the reversal of the direction of the external field[1]:

(i) "Particle-hole" symmetry: $\mathrm{E} \to -E$, $\varphi \to -\varphi$,

(ii) Spatial reflection symmetry: $E \to -E$, $x_\parallel \to -x_\parallel$.

Other terms are expected to be irrelevant in the sense of renormalization group. When $E=0$, the nonlinear term $\propto \varphi^3$ becomes relevant and the anisotropies are removed, so we get back to the model B of Hohenberg and Halperin.[8] When $E \neq 0$, the critical behavior corresponds to the vanishing of either $r_\parallel$ or $r_\perp$, or both.

A renormalized field theory[9] is set up by rewriting the equation of motion into a Martin–Siggia–Rose (MSR) action[10–12]

$$\begin{aligned} A[\varphi, \tilde{\varphi}] = \int d^dx\, dt\, \lambda\tilde{\varphi}\{ [\lambda^{-1}\dot{\varphi} - r_\parallel \partial_\parallel^2 - r_\perp \nabla_\perp^2 \\ + \rho\partial_\parallel^4 + \partial_\parallel^2\nabla_\perp^2 + \eta(\nabla_\perp^2)^2]\, \varphi \\ + (\sigma\partial_\parallel^2 + \nabla_\perp^2)\, \tilde{\varphi} - \tfrac{1}{2}E\, \partial_\parallel \varphi^2 \} \end{aligned} \tag{2.3}$$

where $\tilde{\varphi}(\mathbf{x}, t)$ is a MSR response field.[13] Correlation and response functions (to an external magnetic field) are then path integral averages with weight e^{-A}. In (2.3) we have already chosen the scales of φ and $\tilde{\varphi}$ so that the coefficient of $\tilde{\varphi}\dot{\varphi}$ is one, and those of $\tilde{\varphi}\partial_\parallel^2 \nabla_\perp^2 \varphi$ and $\tilde{\varphi}\nabla_\perp^2 \tilde{\varphi}$ are identical. It is important to realize that $r_\parallel$ and $r_\perp$ can vanish at different temperatures, and therefore we must explore different regions in the parameter space. As explained in Sec. 3, these have different power countings leading to different upper critical dimensions. Thus the RG flows can be quite complicated.

3. RENORMALIZATION—ONE-LOOP CALCULATION

We now describe a renormalization group analysis based on the action (2.3).[14,15] The ultraviolet divergences of the one-particle irreducible vertex functions $\Gamma_{\tilde{N}N}$ with $\tilde{N}$ $\tilde{\varphi}$-legs and N φ-legs are absorbed multiplicatively in redefinitions of the parameters and fields in the action.

It should be emphasized that the vertex functions $\Gamma_{\tilde{\varphi}\tilde{\varphi}}$ and $\Gamma_{\tilde{\varphi}\varphi}$, related linearly to the correlation function G and response function χ respectively, have to be renormalized separately due to the possible absence of the fluctuation-dissipation theorem in the presence of a driving force. However, we observe that the singular parts of G and χ do satisfy the usual form of FDT in the region $r_\parallel > 0$, $r_\perp = 0$. Let us divide the discussion into three parts.

A. $r_\parallel = 0$, $r_\perp > 0$

Power counting in a given diagram shows that the most dominant contribution to ultraviolet divergences comes from regions of integration in which $q_\parallel^2$ is of order $q_\perp$. Relevancy arguments then enable us to drop the terms $\tilde{\varphi}\,\partial_\parallel^2 \nabla_\perp^2 \varphi$, $\tilde{\varphi}(\nabla_\perp^2)^2\varphi$, and $\tilde{\varphi}\nabla_\perp^2\tilde{\varphi}$ in the action. By choosing the scales of φ and $\tilde{\varphi}$ appropriately, we now have

$$A[\varphi,\tilde{\varphi}] = \int d^dx\, dt\, \lambda\tilde{\varphi}\{[\lambda^{-1}\dot{\varphi} - r_\parallel \partial_\parallel^2 - r_\perp \nabla_\perp^2 + \partial_\parallel^4]\,\varphi$$

$$+ \partial_\parallel^2 \tilde{\varphi} - \tfrac{1}{2}E\,\partial_\parallel \varphi^2\} \tag{3.1}$$

Power counting which treats $q_\parallel$ and $q_\perp$ as having distinct dimensions shows that the E-field is relevant with $d_c = 4.5$, and that all other nonlinear couplings are irrelevant with smaller upper critical dimensions. Here only $\Gamma_{\tilde{\varphi}\varphi}$, $\Gamma_{\tilde{\varphi}\tilde{\varphi}}$, and $\Gamma_{\tilde{\varphi}\varphi\varphi}$ are primitively divergent. The divergences are absorbed multiplicatively in redefinitions of the fields and the parameters:

$$\begin{aligned} \varphi_0 &= Z_\varphi^{1/2}\varphi \\ \tilde{\varphi}_0 &= Z_{\tilde{\varphi}}^{1/2}\tilde{\varphi} \\ \lambda_0 &= Z_\lambda \lambda \\ r_{0\perp} &= Z_\perp r_\perp \end{aligned} \tag{3.2}$$

We find that $E_0 \sim q_\perp^{(7-d)/2} q_\parallel^{-5/2} \sim q_\perp^{(9-2d)/4} r_{0\perp}^{-5/8}$ because $q_\parallel$ scales as $q_\perp^{1/2} r_{0\perp}^{1/4}$, which suggests that we define dimensionless parameters e_0 and e as

$$\begin{aligned} e_0 &= r_{0\perp}^{5/8} \Lambda^{-\varepsilon/2} E_0 \\ &= \left(\frac{\kappa}{\Lambda}\right)^{\varepsilon/2} Z_e e \end{aligned} \tag{3.3}$$

where the subscript zero denotes bare quantities. κ is an arbitrary transverse momentum scale, Λ is the $q_\perp$ cutoff, and $\varepsilon = 4.5 - d$. To order one-loop we find (see Appendix I for the evaluations of diagrams involved)

$$\begin{aligned} Z_\perp &= 1 - \frac{5\sqrt{2}}{2^{11}} e^2 \ln\frac{\Lambda}{\kappa} + O(e^4) \\ Z_\varphi &= 1 - \frac{3\sqrt{2}}{2^9} e^2 \ln\frac{\Lambda}{\kappa} + O(e^4) \\ Z_{\tilde{\varphi}} &= Z_\varphi^{-1} \\ Z_e &= Z_\varphi^{-1/2} Z_\perp^{5/8} \\ &= 1 + \frac{23\sqrt{2}}{2^{14}} e^2 \ln\frac{\Lambda}{\kappa} + O(e^4) \\ Z_\lambda &= 1 \end{aligned} \tag{3.4}$$

All the above relations between the Z's are valid to all orders as a consequence of the absence of logarithmic divergences in $\Gamma_{\tilde{\varphi}\varphi\varphi}$, and those proportional to ω and $k_\perp^2$ in $\Gamma_{\tilde{\varphi}\varphi}$ (see Appendix III). The complete cancellation of divergent diagrams of $\Gamma_{\tilde{\varphi}\varphi\varphi}$ is an important property of the interaction vertex $\lambda E\varphi^2\,\partial_\parallel\tilde{\varphi}$, which remains true in all regions considered in this section.

To study the theory away from T_c, we must consider insertions of $\lambda r_\parallel\tilde{\varphi}\partial_\parallel^2\varphi$ in $\Gamma_{\tilde{\varphi}\varphi}$. This leads to one more renormalization constant

$$Z_{\tilde{\varphi}\varphi}=1+\frac{\sqrt{2}}{2^8}e^2\ln\frac{\Lambda}{\kappa}+O(e^4) \tag{3.5}$$

with $(\tilde{\varphi}\varphi)_0=Z_{\tilde{\varphi}\varphi}(\tilde{\varphi}\varphi)$, where $(\tilde{\varphi}\varphi)$ denotes the renormalized composite operator.

B. $r_\parallel>0$, $r_\perp=0$

Here the dominant contribution to ultraviolet divergences comes from the region in which $q_\parallel$ is of order $q_\perp^2$. We can drop $\tilde{\varphi}\partial_\parallel^2\nabla_\perp^2\varphi$, $\tilde{\varphi}\partial_\parallel^4\varphi$, and $\tilde{\varphi}\partial_\parallel^2\tilde{\varphi}$ in the action (2.3) to get

$$\begin{aligned}A[\varphi,\tilde{\varphi}]=\int d^dx\,dt\,\lambda\tilde{\varphi}\{&[\lambda^{-1}\dot{\varphi}-r_\parallel\partial_\parallel^2-r_\perp\nabla_\perp^2+(\nabla_\perp^2)^2]\,\varphi\\&+\nabla_\perp^2\tilde{\varphi}-\tfrac{1}{2}E\,\partial_\parallel\varphi^2\}\end{aligned} \tag{3.6}$$

The coupling to E has $d_c=5$ here, and only $\Gamma_{\tilde{\varphi}\varphi}$ and $\Gamma_{\tilde{\varphi}\varphi\varphi}$ are primitively divergent. Defining Z's as in (3.2) (except $r_\perp$), together with

$$\begin{aligned}r_{0\parallel}&=Z_\parallel r_\parallel\\e_0&=r_{0\parallel}^{-3/4}\Lambda^{-\varepsilon/2}E_0\\&=\left(\frac{\kappa}{\Lambda}\right)^{\varepsilon/2}Z_e e\end{aligned} \tag{3.7}$$

where $\varepsilon=5-d$, we find to order one-loop (see Appendix I)

$$\begin{aligned}Z_\parallel&=1-\frac{3}{32}e^2\ln\frac{\Lambda}{\kappa}+O(e^4)\\Z_\varphi&=Z_{\tilde{\varphi}}^{-1}=1\\Z_\lambda&=1\\Z_e&=Z_\parallel^{-3/4}\\&=1+\frac{9}{2^7}e^2\ln\frac{\Lambda}{\kappa}+O(e^4)\end{aligned} \tag{3.8}$$

Again the relations between the Z's above are valid to all orders as a result of the absence of divergences in $\Gamma_{\tilde{\varphi}\varphi\varphi}$, in $\Gamma_{\tilde{\varphi}\tilde{\varphi}}$, and of any proportional to ω and $k_\perp^4$ in $\Gamma_{\tilde{\varphi}\varphi}$ (see Appendix III). There is no further UV divergence associated with insertions of $r_\perp \tilde{\varphi}\nabla_\perp^2 \varphi$ in $\Gamma_{\tilde{N}N}$.

C. Near $r_\parallel = 0$, $r_\perp = 0$

In this region the full action of (2.3) is needed. To simplify the evaluation of integrals, one may calculate with zero coefficient of the term $\tilde{\varphi}(\nabla_\perp^2)^2\varphi$, namely $\eta_0 = 0$, because $k_\perp^4$ will not be generated by the perturbation expansion of $\Gamma_{\tilde{\varphi}\varphi}$. Stability analysis then shows that $\eta_0 = 0$ is a stable fixed point.

In this region $d_c = 8$, and only $\Gamma_{\tilde{\varphi}\varphi}$, $\Gamma_{\tilde{\varphi}\tilde{\varphi}}$, and $\Gamma_{\tilde{\varphi}\varphi\varphi}$ are primitively divergent. We define renormalization constants as

$$\begin{aligned} E_0 &= \Lambda^{\varepsilon/2}\rho_0^{-3/4} e_0 \\ e_0 &= \left(\frac{\kappa}{\Lambda}\right)^{\varepsilon/2} Z_e e \\ \rho_0 &= Z_\rho \rho \\ \sigma_0 &= Z_\sigma \sigma \end{aligned} \tag{3.9}$$

where $\varepsilon = 8 - d$. To order one-loop we find

$$\begin{aligned} Z_\lambda &= 1 - \frac{5}{2^9}(13w - 37)\, e^2 \ln\frac{\Lambda}{\kappa} + O(e^4) \\ Z_\rho &= 1 + \frac{1}{2^8}(317\text{–}45w)\, e^2 \ln\frac{\Lambda}{\kappa} + O(e^4) \\ Z_\sigma &= 1 - \frac{1}{2^6} w^{-1}(63\text{–}70w + 15w^2)\, e^2 \ln\frac{\Lambda}{\kappa} + O(e^4) \\ Z_e &= Z_\lambda^{-3/2} Z_\rho^{3/4} \\ &= 1 + \frac{3}{2^8}(5w + 33)\, e^2 \ln\frac{\Lambda}{\kappa} + O(e^4) \end{aligned} \tag{3.10}$$

where $w = \sigma/\rho$.

A given physical system may exhibit different critical behaviors governed by fixed points in different regions of the parameter space, depending upon the trajectory of the physical system as external parameters like temperature T vary. We are interested in the critical behavior as T is lowered, starting in a high-temperature disordered phase. If the driving force is small, then it is sufficient to calculate the difference of $r_\parallel$ and $r_\perp$ to the lowest order in E in order to determine the trajectory of such a physical system. This tells us which fixed point is appropriate. Straightforward calculations show that the interaction vertex modifies only the parallel parameter as

$$r_{\|} = r_{0\|} + C_d E_0^2 + O(E_0^4) \tag{3.11a}$$

$$r_{\perp} = r_{0\perp} \tag{3.11b}$$

where the d-dependent constant $C_d > 0$ for all dimensions of interest. Hence $r_{\perp}$ vanishes first as T is lowered. The critical behavior is then determined by the fixed point in the region $r_{\|} > 0$, $r_{\perp} = 0$ with $d_c = 5$. This is consistent with the finding of Gawedzki and Kupiainen[5] using a supersymmetric formulation.

Concerning the change of T_c in the presence of E, all we have shown above is that the effect of the fluctuations caused by E alone leaves the mean field critical temperature unchanged to all orders in E because $r_{\perp}$ vanishes at a higher temperature and it does not get renormalized. To obtain $T_c(E)$, we would need to consider the interactions between the fluctuations caused by E and those arising from nonlinear coupling of strength g among φ's. In a perturbation theory, such interactions are represented to the lowest order by two-loop diagrams of order $E^2 g$.

4. CALLAN–SYMANZIK EQUATIONS AND CRITICAL EXPONENTS

The Callan–Symanzik (C–S) equations[9] are derived from the independence of renormalized vertex functions or Green's functions on the cutoff Λ. As an example, in the region $r_{\|} = 0$, $r_{\perp} > 0$, $G_{\varphi\varphi}$ obeys

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta \frac{\partial}{\partial e_0} + \gamma - \zeta_{\perp} r_{0\perp} \frac{\partial}{\partial r_{0\perp}} - \mu r_{0\|} \frac{\partial}{\partial r_{0\|}}\right) \cdot G_{\varphi\varphi}(k, \omega, r_{o\|}, r_{0\perp}, E_0, \lambda, \Lambda) = 0 \tag{4.1}$$

The renormalization group functions

$$\beta = \Lambda \left(\frac{\partial}{\partial \Lambda} e_0\right)_{\text{ren}}$$

$$\gamma = -\Lambda \left(\frac{\partial}{\partial \Lambda} \ln Z_{\varphi}\right)_{\text{ren}}$$

$$\zeta_{\perp} = -\Lambda \left(\frac{\partial}{\partial \Lambda} \ln Z_{\perp}\right)_{\text{ren}}$$

$$\mu = -\Lambda \left(\frac{\partial}{\partial \Lambda} \ln Z_{\tilde{\varphi}\varphi}\right)_{\text{ren}} \tag{4.2}$$

are obtained by differentiating at fixed renormalized parameters. The zeroes e_0^* of $\beta(e_0)$ define the fixed points. By dimensional analysis

$$G_{\varphi\varphi} = \lambda_0^{-1} r_{0\perp}^{1/2} \Lambda^{-3} f\left(\frac{\omega}{\lambda\Lambda^2}, \frac{k_\perp}{\Lambda}, \frac{k_\parallel}{r_{0\perp}^{1/4}\Lambda^{1/2}}, e_0, \frac{r_{0\parallel} r_{0\perp}^{1/2}}{\Lambda}\right) \tag{4.3}$$

which enables us to rewrite the C–S equation as

$$\left[3 - \gamma + \frac{1}{2}\zeta_\perp + k_\perp \frac{\partial}{\partial k_\perp} + \left(\frac{1}{2} + \frac{1}{4}\zeta_\perp\right) k_\parallel \frac{\partial}{\partial k_\parallel} + 2\omega \frac{\partial}{\partial \omega}\right.$$
$$\left. + \left(1 + \frac{1}{2}\zeta_\perp + \mu\right) r_{0_\parallel} \frac{\partial}{\partial r_{0_\parallel}} + \beta \frac{\partial}{\partial e_0}\right] G_{\varphi\varphi} = 0 \tag{4.4}$$

whose solution gives the scaling form of $G_{\varphi\varphi}$ from which critical exponents are identified.

Due to anisotropies caused by the driving force, a suitable set of critical exponents for all regions of the parameter space consists of $\eta_\parallel$, $\eta_\perp$, $\eta_\parallel{}^{(\mathrm{RS})}$, $\eta_\perp{}^{(\mathrm{RS})}$, $\nu_\parallel$, $\nu_\perp$, $z_\parallel$, $z_\perp$, β, γ and a new exponent, denoted as ψ, to describe the possible violation of the fluctuation–dissipation theorem. Here (RS) stands for "real space." These exponents are defined via the scaling forms of $G_{\varphi\varphi}$.

$$\begin{aligned} G_{\varphi\varphi}(\mathbf{k}, t) &= \int \frac{d\omega}{2\pi} e^{-i\omega t} G_{\varphi\varphi}(\mathbf{k}, \omega) \\ &= k_\perp^{-2+\eta_\perp} f_\perp(k_\parallel k_\perp^{-(2-\eta_\perp)/(2-\eta_\parallel)}, k_\perp^{z_\perp} t) \\ &= k_\parallel^{-2+\eta_\parallel} f_\parallel(k_\perp k_\parallel^{-(2-\eta_\parallel)/(2-\eta_\perp)}, k_\parallel^{z_\parallel} t) \end{aligned} \tag{4.5}$$

defines $\eta_\parallel$, $\eta_\perp$, $z_\parallel$, and $z_\perp$;

$$\begin{aligned} G_{\varphi\varphi}(\mathbf{k}, \omega, r) &= k_\perp^{-2+\eta_\perp - z_\perp} f_\perp(k_\perp \xi_\perp, k_\parallel \xi_\parallel, \omega) \\ \xi_\perp &\sim r^{-\nu_\perp} \\ \xi_\parallel &\sim r^{-\nu_\parallel} \end{aligned} \tag{4.6}$$

define $\nu_\perp$ and $\nu_\parallel$. Here r stands for the appropriate reduced temperature variable. The response function χ defines γ:

$$\begin{aligned} \chi(\mathbf{k}, \omega) &= \lambda_0(k_\parallel^2 + k_\perp^2)\, G_{\tilde{\varphi}\varphi}(\mathbf{k}, \omega) \\ &= r^{-\gamma} f(k_\perp \xi_\perp, k_\parallel \xi_\parallel, \omega) \\ &\quad + \text{less singular term} \end{aligned} \tag{4.7}$$

and the nonvanishing of ψ reflects the breakdown of the usual form of FDT, i.e., $G_{\varphi\varphi}(\mathbf{k}, \omega) = (2/\omega) \operatorname{Im} \chi(\mathbf{k}, \omega)$:

$$\gamma = \nu_\perp (2 - \eta_\perp + \psi) \tag{4.8}$$

Fourier transforming $G(\mathbf{k}, t)$ to real space, we find that the conventionally defined real-space correlation function exponents $\eta_\parallel^{(\mathrm{RS})}$ and $\eta_\perp^{(\mathrm{RS})}$ are not the same as the corresponding momentum-space exponents. For simplicity, consider the equal-time correlation function

$$\begin{aligned} G(\mathbf{x}, t=0) &= \int \frac{1}{(2\pi)^d} dk_\parallel \, d^{d-1}k_\perp \, e^{ik_\parallel x_\parallel + i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} \\ &\quad \times G(\mathbf{k}, t=0) \\ &= \frac{\xi_\perp}{\xi_\parallel} \cdot \xi_\perp^{-(d-2+\eta_\perp)} g\left(\frac{x_\perp}{\xi_\perp}, \frac{x_\parallel}{\xi_\parallel}\right) \end{aligned} \tag{4.9}$$

Comparing this with the definitions of $\eta_\parallel^{(\mathrm{RS})}$ and $\eta_\perp^{(\mathrm{RS})}$

$$\begin{aligned} G(x_\parallel \neq 0, x_\perp = 0, t=0, r=0) &\sim \frac{1}{x_\parallel^{d-2+\eta_\parallel^{(\mathrm{RS})}}} \\ G(x_\parallel = 0, x_\perp \neq 0, t=0, r=0) &\sim \frac{1}{x_\perp^{d-2+\eta_\perp^{(\mathrm{RS})}}} \end{aligned} \tag{4.10}$$

yields

$$\begin{aligned} \eta_\parallel^{(\mathrm{RS})} &= \eta_\parallel - (d-1) \cdot \frac{\eta_\parallel - \eta_\perp}{2 - \eta_\perp} \\ \eta_\perp^{(\mathrm{RS})} &= \eta_\perp + \frac{\eta_\parallel - \eta_\perp}{2 - \eta_\parallel} \end{aligned} \tag{4.11}$$

As long as $\eta_\parallel = \eta_\perp \equiv \eta$, we recover $\eta_\parallel^{(\mathrm{RS})} = \eta_\perp^{(\mathrm{RS})} = \eta$.

Other scaling relations follow directly from the definitions of exponents:

$$\frac{\nu_\perp}{\nu_\parallel} = \frac{2 - \eta_\parallel}{2 - \eta_\perp} \tag{4.12}$$

$$\frac{z_\perp}{z_\parallel} = \frac{2 - \eta_\perp}{2 - \eta_\parallel} \tag{4.13}$$

$$\gamma = \nu_\parallel \left(2 - \eta_\parallel + \frac{2 - \eta_\parallel}{2 - \eta_\perp} \psi\right) \tag{4.14}$$

Hence one can choose an independent set of exponents for this model as $\{\eta_{\|}, \eta_{\perp}, z_{\perp}, v_{\perp}, \psi\}$. Notice that these scaling relations become the conventional ones once the anisotropy is removed.

There are additional scaling relations specific to the fixed point in each region of the parameter space. This is a direct consequence of the absence of some of the ultraviolet divergences in the vertex functions, which leads to the relations among Z's in (3.4) and (3.8). Let us divide our discussions into two parts:

A. $r_{\|}=0,\ r_{\perp}>0$

By definitions (4.2), we obtain

$$\left.\begin{aligned}\gamma(e_0)&=\frac{3\sqrt{2}}{2^9}e_0^2+O(e_0^4)\\ \zeta_{\perp}(e_0)&=\frac{5\sqrt{2}}{2^{11}}e_0^2+O(e_0^4)\\ \mu(e_0)&=-\frac{\sqrt{2}}{2^8}e_0^2+O(e_0^4)\end{aligned}\right\}\tag{4.15}$$

$$\begin{aligned}\beta(e_0)&=\frac{1}{2}e_0\left[-\varepsilon+\gamma(e_0)-\frac{5}{4}\zeta_{\perp}(e_0)\right]\\ &=\frac{1}{2}e_0\left[-\varepsilon+\frac{23\sqrt{2}}{2^{13}}e_0^2\right]+O(e_0^5)\end{aligned}\tag{4.16}$$

The infrared stable fixed point for $\varepsilon=4.5-d>0$ is thus

$$e_0^{*2}=\frac{2^{13}}{23\sqrt{2}}\varepsilon+O(\varepsilon^2)\tag{4.17}$$

Critical exponents identified from the solution of C–S equations are given in terms of the RG functions evaluated at the fixed point:

$$\begin{aligned}\eta_{\perp}&=1-\tfrac{1}{2}\zeta_{\perp}^*+\gamma^*\\ \eta_{\|}&=4\cdot\frac{\gamma^*-\zeta_{\perp}^*}{2-\zeta_{\perp}^*}\\ z_{\perp}&=2\\ v_{\perp}&=(1+\tfrac{1}{2}\zeta_{\perp}^*+\mu^*)^{-1}\\ \psi&=\gamma^*\end{aligned}\tag{4.18}$$

These together with $\gamma^* = \varepsilon + \frac{5}{4}\zeta_\perp^*$, which follows from the β-function, give three more scaling relations in this region, relating $\eta_\perp$, $\eta_\parallel$, $\nu_\perp$, and ψ as

$$\begin{aligned} \eta_\parallel &= 2\,\frac{2\varepsilon - 1 + \eta_\perp}{5 + 2\varepsilon - 2\eta_\perp} \\ z_\parallel &= \frac{12}{5 + 2\varepsilon - 2\eta_\perp} \\ \psi &= \tfrac{1}{3}(5\eta_\perp - 2\varepsilon - 5) \end{aligned} \tag{4.19}$$

These are valid to all orders in ε. Only two exponents are then independent at this fixed point, for example:

$$\begin{aligned} \eta_\perp &= 1 + \tfrac{38}{23}\,\varepsilon + O(\varepsilon^2) \\ \nu_\perp &= 1 + \tfrac{22}{23}\varepsilon + O(\varepsilon^2) \end{aligned} \tag{4.20}$$

B. $r_\parallel > 0$, $r_\perp = 0$

Here as shown in 3B, $Z_\varphi = Z_{\tilde{\varphi}} = Z_\lambda = Z_{\tilde{\varphi}\varphi} = 1$, and the C–S equation is simpler:

$$\left[\Lambda\,\frac{\partial}{\partial\Lambda} + \beta\,\frac{\partial}{\partial e_0} - \zeta_\parallel r_{0_\parallel}\,\frac{\partial}{\partial r_{0_\parallel}}\right] G_{\varphi\varphi} = 0 \tag{4.21}$$

where

$$\zeta_\parallel(e_0) = -\Lambda\left(\frac{\partial}{\partial\Lambda}\ln Z_\parallel\right)_{\text{ren}} \tag{4.22}$$

$$\beta(e_0) = \tfrac{1}{2}e_0(-\varepsilon + \tfrac{3}{2}\zeta_\parallel(e_0)) \tag{4.23}$$

At the infrared stable fixed point e_0^* for $\varepsilon = 5 - d > 0$,

$$\zeta_\parallel^* = \frac{2\varepsilon}{3} \tag{4.24}$$

to all orders, which is identical to the O(1-loop) result. Since all exponents are expressed in terms of $\zeta_\parallel^*$, we get expressions to all orders in ε:

$$\begin{aligned} \eta_\parallel &= 2 - \frac{4}{4 + \zeta_\parallel^*} \\ &= 2\cdot\frac{3 + \varepsilon}{6 + \varepsilon} \\ \eta_\perp &= 0 \\ z_\perp &= 4 \\ \nu_\perp &= \tfrac{1}{2} \\ \psi &= 0 \end{aligned} \tag{4.25}$$

The scaling relations (4.11)–(4.14) then yield other exponents:

$$\eta_{\parallel}^{(\mathrm{RS})} = \frac{(\varepsilon - 2)(3 + \varepsilon)}{6 + \varepsilon}$$

$$\eta_{\perp}^{(\mathrm{RS})} = 1 + \frac{\varepsilon}{3}$$

$$z_{\parallel} = \frac{12}{6 + \varepsilon}$$

$$\nu_{\parallel} = 1 + \frac{\varepsilon}{6}$$

$$\gamma = 1 \tag{4.26}$$

The calculation of the exponent β involves the consideration of the dangerous irrelevant variable g which would couple to a term $\partial_{\perp}^{2} \phi^{3}$ in (2.2a). In Appendix IV we show that $\beta = \frac{1}{2}$ for all $d > 2$.

The fact that we are able to obtain results to all orders is a direct consequence of the absence of logarithmic divergences in all vertex functions except $\Gamma_{\tilde{\varphi}\varphi}$. We have checked that to O(1-loop) $G_{\varphi\varphi}(\mathbf{k}, \omega)$ and $(2/\omega) \operatorname{Im} \chi(\mathbf{k}, \omega)$ are not equal but that the leading singular parts are, which is indicated by the vanishing of the new exponent ψ.

We observe that all transverse exponents and γ are of mean field values, and the following inequalities hold for all $\varepsilon > 0$:

$$\eta_{\parallel}^{(\mathrm{RS})} < \eta_{\perp}^{(\mathrm{RS})} \tag{4.27}$$

$$\nu_{\parallel} > \nu_{\perp} \tag{4.28}$$

$$z_{\parallel} < z_{\perp} \tag{4.29}$$

(4.27) and (4.28) imply that the spatial correlation is elongated along the direction of the driving force, which is consistent with computer simulation results[1] in $d = 2$ and 3. (4.29) implies the faster temporal decay of longitudinal fluctuations, to be compared with anisotropic decays in low dimensions above T_c.[3,4]

5. SUMMARY AND FURTHER REMARKS

In this paper we have employed a field theoretic method to study the critical behavior of driven diffusive systems. The constant uniform driving force induces spatial anisotropies. To describe the properties of the systems we need a relatively large parameter space, including a longitudinal and a transverse reduced temperature parameter. The RG flows in such parameter space become quite complicated. Simple lowest-order calculations of the shift of $r_{\|}$ due to the anisotropy induced by E show that $r_{\perp}$ vanishes first as T is lowered from above T_c. The critical behavior of the system is thus governed by an infrared stable fixed point below $d=5$. Above $d=5$, it is determined by the Gaussian fixed point. In that case, although the fluctuations are irrelevant, they still make the correlations anisotropic, leading to the exponents given by (4.25) and (4.26) with $\varepsilon=0$. This is not equivalent to mean field theory, which ignores all effect of the fluctuations.

The surprising result is that at this fixed point one is able to obtain expressions for the exponents to all orders in the perturbation expansion, without evaluating a single Feynman diagram. This follows from the absence of all ultraviolet divergences in vertex functions except $\Gamma_{\tilde{\varphi}\varphi}$.

To describe the anisotropic critical properties, we find it necessary to introduce longitudinal as well as transverse exponents, and a new exponent ψ to take care of the possible breakdown of the conventional form of FDT. ψ turns out to be zero at the above-mentioned fixed point, indicating that the most singular parts do satisfy FDT, though the complete expressions do not.

Janssen and Schmittmann[4] showed by field theoretic techniques that such driven diffusive systems obey detailed balance symmetry[12] when they are above the critical point. We are, however, not able to rewrite our MSR action in the detailed-balance symmetric form whenever $r_{\|}$ or $r_{\perp}$ vanish separately. We therefore conjecture that detailed balance symmetry is broken in connection with the breakdown of FDT at the critical point, since FDT follows from the detailed-balance symmetric form of the equation of motion.[16] There have also been discussions on the connection between supersymmetry, microscopic reversibility, and FDT associated with Langevin equations,[17,18] and it is very likely that DB, FDT, and SUSY are intimately related. This question requires further elucidation.

The elongated correlation along the direction of the driving force found at the $r_{\perp}=0$ fixed point agrees qualitatively with computer simulations, though quantitative comparisons are not yet available. Precise measurements of critical exponents by simulations are very desirable.

The fact that the two reduced temperature variables can vanish at different temperatures may give rise to the possibility of different ordered phases as T is lowered further below T_c. However, to understand this we have to first study the properties of the ordered phase just below T_c.

The field theoretic method used in this paper is systematic, yielding scaling forms of correlation functions and scaling relations between exponents. It should be possible to study more complicated systems with different symmetries of driving forces[5] and with different conservation laws.[8] Apparently these properties at least partially characterize the universality classes[8] of critical systems maintained in nonequilibrium steady states.

APPENDIX

I. Feynman Diagrams

Here the Feynman rules and the one-loop Feynman diagrams in the regions $r_{\parallel}=0$, $r_{\perp}>0$; $r_{\parallel}>0$, $r_{\perp}=0$; and $r_{\perp}=0$, $r_{\parallel}=0$ are presented.

There are two kinds of propagators. The response propagator $G^0_{\tilde{\varphi}\varphi}(\mathbf{k}, \omega)$ and the correlation propagator $G^0_{\varphi\varphi}(\mathbf{k}, \omega)$ as shown in Fig. 1. The interaction vertex corresponding to the driving force and the composite operator vertex are shown in Fig. 2.

In the integrals presented below, the appropriate reduced temperature variable(s) is set to zero in the integrands (i.e., "massless theory"[10]) and they are evaluated at ($k_{\parallel}\neq 0$, $k_{\perp}=0$, $\omega=0$). A factor of $\Omega_{d-1}/(2\pi)^{d-1}$ has been absorbed into a redefinition of E_0^2, where Ω_d = surface area of a d-dimensional sphere. From now on the subscript zero for bare parameters is suppressed for brevity; all parameters are understood to be bare.

A. $r_{\parallel}=0$, $r_{\perp}>0$

The propagators are

$$
\begin{aligned}
G^0_{\tilde{\varphi}\varphi}(\mathbf{k}, \omega) &= \frac{1}{-i\omega+\lambda(r_{\perp}k_{\perp}^2+k_{\parallel}^4)} \\
G^0_{\varphi\varphi}(\mathbf{k}, \omega) &= \frac{2\lambda k_{\parallel}^2}{\omega^2+\lambda^2(r_{\perp}k_{\perp}^2+k_{\parallel}^4)^2}
\end{aligned}
\tag{A.1}
$$

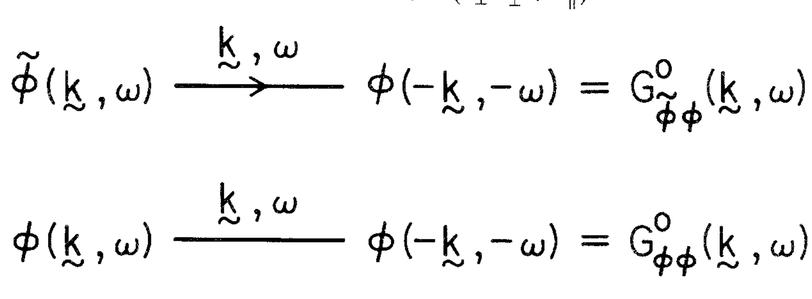

Fig. 1. The diagrammatic representations of the bare propagators of the perturbation theory. $G^0_{\tilde{\varphi}\varphi}$ is the response line with arrow pointing away from $\tilde{\varphi}$; $G^0_{\varphi\varphi}$ is the correlation line. There is no $G_{\tilde{\varphi}\tilde{\varphi}}$ propagator.

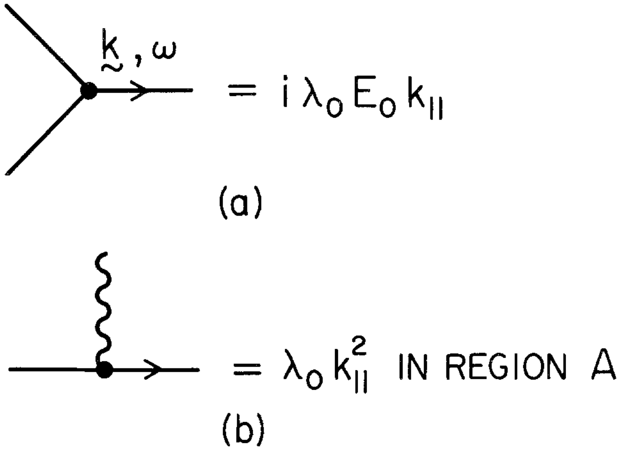

Fig. 2. (a) The three-point vertex which represents the coupling of φ to E; (b) the two-point vertex which represents the composite operator $\lambda_0 r_{0_\parallel} \tilde{\varphi}\, \partial_\parallel^2 \varphi$.

In this region, $d_c = 4.5$. There are three primitively divergent vertex functions. We denote $[1/(2\pi)^d]\int_{-\infty}^{\infty} dq_\parallel \int^\wedge d^{d-1}q_\perp$ by $\int^\wedge$. To order 1-loop, we find after, integrating over the internal frequencies (see Fig. 3),

$$\begin{aligned}
\Gamma_{\tilde{\varphi}\varphi}: I_{a,A} &= -\lambda E^2 k_\parallel \int^\wedge \frac{q_\parallel^2 (k-q)_\parallel}{r_\perp q_\perp^2 + q_\parallel^4} \\
&\quad \times \frac{1}{2 r_\perp q_\perp^2 + q_\parallel^4 + (k-q)_\parallel^4} \\
&= -\lambda k_\parallel^4 e_0^2 \left(\frac{\Lambda r_\perp^{1/2}}{k_\parallel^2}\right)^\varepsilon \\
&\quad \times \left[\frac{\sqrt{2}}{2^7}\left(\frac{\Lambda r_\perp^{1/2}}{k_\parallel^2}\right) - \frac{5\sqrt{2}}{2^{11}} \ln\left(\frac{\Lambda r_\perp^{1/2}}{k_\parallel^2}\right)\right] \\
&\quad + O(\Lambda^{-1})
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\Gamma_{\tilde{\varphi}\tilde{\varphi}}: I_{b,A} &= -\lambda E^2 k_\parallel^2 \int^\wedge \frac{q_\parallel^2}{r_\perp q_\perp^2 + q_\parallel^4} \frac{(k-q)_\parallel^2}{r_\perp q_\perp^2 + (k-q)_\parallel^4} \\
&\quad \times \frac{1}{2 r_\perp q_\perp^2 + q_\parallel^4 + (k-q)_\parallel^4} \\
&= -\lambda k_\parallel^2 e_0^2 \left(\frac{\Lambda r_\perp^{1/2}}{k_\parallel^2}\right)^\varepsilon \\
&\quad \times \frac{3\sqrt{2}}{2^8} \ln\left(\frac{\Lambda r_\perp^{1/2}}{k_\parallel^2}\right) + O(\Lambda^{-1})
\end{aligned} \tag{A.3}$$

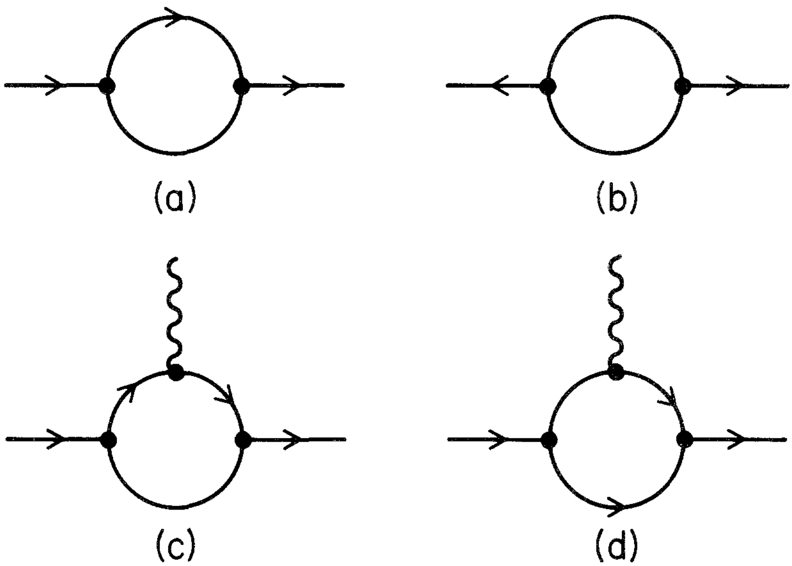

Fig. 3. (a) The one-loop diagram of $\Gamma_{\tilde{\varphi}\varphi}$, which corresponds to an integral I_a; (b) the one-loop diagram of $\Gamma_{\tilde{\varphi}\tilde{\varphi}}$, which corresponds to an integral I_b; (c, d) the one-loop diagrams of $\Gamma_{\tilde{\varphi}\varphi,1}$, i.e., $\Gamma_{\tilde{\varphi}\varphi}$, with one insertion of the composite operator of Fig. 2b. The corresponding integrals are I_c and I_d.

$$\Gamma_{\tilde{\varphi}\varphi,1}: I_c = -\lambda E^2 k_{\parallel} \int^{\Lambda} \frac{q_{\parallel}^2}{r_{\perp} q_{\perp}^2 + q_{\parallel}^4} \frac{(k-q)_{\parallel}^3}{[2r_{\perp} q_{\perp}^2 + q_{\parallel}^4 + (k-q)_{\parallel}^4]^2}$$

$$= -\lambda k_{\parallel}^2 e_0^2 \left(\frac{\Lambda r_{\perp}^{1/2}}{k_{\parallel}^2}\right)^{\varepsilon} \frac{\sqrt{2}}{2^7} \ln\left(\frac{\Lambda r_{\perp}^{1/2}}{k_{\parallel}^2}\right) + O(\Lambda^{-1}) \tag{A.4}$$

$$\Gamma_{\tilde{\varphi}\varphi,1}: I_d = -\lambda E^2 k_{\parallel} \int^{\Lambda} \frac{q_{\parallel}^2}{r_{\perp} q_{\perp}^2 + q_{\parallel}^4} \frac{(k-q)_{\parallel} q_{\parallel}^2}{[2r_{\perp} q_{\perp}^2 + q_{\parallel}^4 + (k-q)_{\parallel}^4]^2}$$

$$= -\tfrac{1}{2} I_c \tag{A.5}$$

Here we denote $\Gamma_{\tilde{\varphi}\varphi,L}$ as the vertex function $\Gamma_{\tilde{\varphi}\varphi}$ with L insertions of composite operator $\tilde{\varphi}\varphi$.

B. $r_{\parallel} > 0$, $r_{\perp} = 0$

The propagators are

$$\begin{aligned} G^0_{\tilde{\varphi}\varphi}(\mathbf{k}, \omega) &= \frac{1}{-i\omega + \lambda(r_{\parallel} k_{\parallel}^2 + k_{\parallel}^4)} \\ G^0_{\varphi\varphi}(\mathbf{k}, \omega) &= \frac{2\lambda k_{\perp}^2}{\omega^2 + \lambda^2 (r_{\parallel} k_{\parallel}^2 + k_{\perp}^4)^2} \end{aligned} \tag{A.6}$$

Here $d_c = 5$. There is only one primitively divergent vertex function: $\Gamma_{\tilde{\varphi}\varphi}$. To order 1-loop we find for the same diagram as Fig. 3.a:

$$I_{a,B} = -\lambda E^2 k_\| \int^\wedge \frac{q_\perp^2}{r_\| q_\|^2 + q_\perp^4} \frac{(k-q)_\|}{2q_\perp^4 + r_\| q_\|^2 + r_\|(k-q)_\|^2}$$
$$= -\lambda r_\| k_\|^2 e_0^2 \left(\frac{\Lambda}{r_\|^{1/4} k_\|^{1/2}}\right)^\varepsilon \frac{3}{32} \ln\left(\frac{\Lambda}{r_\|^{1/4} k_\|^{1/2}}\right) + O(\Lambda^{-1}) \tag{A.7}$$

C. $r_\| = 0$, $r_\perp = 0$

The propagators are

$$G^0_{\tilde{\varphi}\varphi}(\mathbf{k}, \omega) = \frac{1}{-i\omega + \lambda k_\|^2(\rho k_\|^2 + k_\perp^2)}$$
$$G^0_{\varphi\varphi}(\mathbf{k}, \omega) = \frac{2\lambda(\sigma k_\|^2 + k_\perp^2)}{\omega^2 + \lambda^2 k_\|^4(\rho k_\|^2 + k_\perp^2)^2} \tag{A.8}$$

In this region $d_c = 8$. The primitively divergent vertex functions are $\Gamma_{\tilde{\varphi}\varphi}$, $\Gamma_{\tilde{\varphi}\tilde{\varphi}}$, and $\Gamma_{\tilde{\varphi}\varphi\varphi}$. To order 1-loop we find (see Fig. 3)

$$\Gamma_{\tilde{\varphi}\varphi}: I_{a,C} = -\lambda E^2 k_\| \int^\wedge \frac{\sigma q_\|^2 + q_\perp^2}{\rho q_\|^4 + q_\|^2 q_\perp^2}$$
$$\times \frac{(k-q)_\|}{\rho q_\|^4 + q_\|^2 q_\perp^2 + \rho(k-q)_\|^4 + (k-q)_\|^2 (k-q)_\perp^2}$$
$$= \lambda k_\|^2 (w-5) \left(\frac{\Lambda}{k_\| \rho^{1/2}}\right)^\varepsilon e_0^2 \frac{3}{2^6} \Lambda^2$$
$$- \lambda k_\|^4 (155w - 819) \left(\frac{\Lambda}{k_\| \rho^{1/2}}\right)^\varepsilon e_0^2 \frac{1}{2^9} \ln \frac{\Lambda}{k_\| \rho^{1/2}}$$
$$- \lambda k_\|^2 k_\perp^2 (13w - 37) \left(\frac{\Lambda}{k_\| \rho^{1/2}}\right)^\varepsilon e_0^2 \frac{5}{2^9} \ln \frac{\Lambda}{k_\| \rho^{1/2}}$$
$$+ O(\Lambda^{-2}) \tag{A.9}$$

$$\Gamma_{\tilde{\varphi}\tilde{\varphi}}: I_{b,C} = -\lambda E^2 k_\|^2 \int^\wedge \frac{\sigma q_\|^2 + q_\perp^2}{\rho q_\|^4 + q_\|^2 q_\perp^2} \frac{\sigma(k-q)_\|^2 + q_\perp^2}{\rho(k-q)_\|^4 + (k-q)_\|^2 q_\perp^2}$$
$$\times \frac{1}{\rho q_\|^4 + q_\|^2 q_\perp^2 + \rho(k-q)_\|^4 + (k-q)_\|^2 q_\perp^2}$$
$$= -\lambda k_\|^2 \rho(-63 + 70w - 15w^2) \left(\frac{\Lambda}{k_\| \rho^{1/2}}\right)^\varepsilon e_0^2$$
$$\times \frac{1}{2^5} \ln \frac{\Lambda}{k_\| \rho^{1/2}} + O(\Lambda^{-2}) \tag{A.10}$$

These results lead to the expressions for the renormalization constants Z in Sec. 3.

II. Crossover Exponent

The crossover exponent φ from the $E=0$ fixed point (the Wilson–Fisher fixed point) to the $E\neq 0$ fixed point is given by the ratio of the eigenvalue y_E associated with E and the thermal eigenvalue y_T, both calculated at the Wilson–Fisher fixed point in an $\varepsilon=4-d$ expansion. Evaluating the integral corresponding to Fig. 4 gives y_E:

$$y_E=2+\frac{\varepsilon}{6}+O(\varepsilon^2)$$

As (A.11)

$$Y_T=2-\frac{\varepsilon}{3}+O(\varepsilon^2)$$

to the lowest order in $\varepsilon=4-d$

$$\varphi=\frac{y_E}{y_T}=1+\frac{\varepsilon}{4}+O(\varepsilon^2) \tag{A.12}$$

Consequently the correlation functions etc. depend for small E on the variable $Er^{-\varphi}$. This determines the shape of the phase boundary near $E=0$ in a standard way.[9]

III. Absence of Certain Ultraviolet Divergences

Here we consider the absence of UV divergences in some of the vertex functions to all orders. We will confine our discussion to the region $r_{\parallel}>0$, $r_{\perp}=0$ only, as a similar argument holds for other regions.

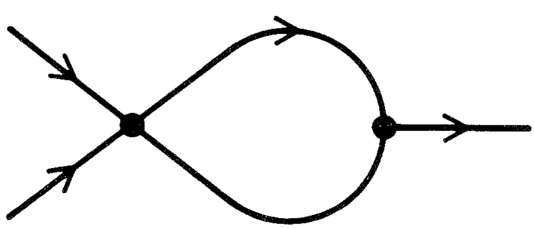

Fig. 4. The one-loop diagram in the calculation of the crossover exponent φ.

Consider the two-point function

$$\Gamma_{\tilde{\varphi}\varphi}(\mathbf{k}, \omega) = -i\omega + \lambda(r_{\parallel} k_{\parallel}^2 + k_{\perp}^4) - \Sigma_{\tilde{\varphi}\varphi}(\mathbf{k}, \omega) \tag{A.13}$$

where $\Sigma_{\tilde{\varphi}\varphi}(\mathbf{k}, \omega)$ denotes the sum of all one-particle irreducible (1PI) diagrams with one incoming and one outgoing leg. The three-point vertex of Fig. 2(a) which attaches to the outgoing leg gives a factor of $k_{\parallel}$ to $\Sigma_{\tilde{\varphi}\varphi}$. It can be shown that the number of internal response lines (Fig. 1) is odd for all the 1PI diagrams; therefore the integrals must all vanish at $k_{\parallel} = 0$ by the oddness of the integrands. Power counting then determines that, at $d = 5$,

$$\Sigma_{\tilde{\varphi}\varphi}(\mathbf{k}, \omega) \sim k_{\parallel}(k_{\parallel} \ln \Lambda + O(\Lambda^{-1})) \tag{A.14}$$

where terms of the form $(\omega/k_{\parallel}) \ln \Lambda$, $(k_{\perp}^4/k_{\parallel}) \ln \Lambda$, and $k_{\perp}^2 \ln \Lambda$ inside the brackets must be absent by the vanishing of the integrals at $k_{\parallel} = 0$ and finite ω and $k_{\perp}$. Consequently $Z_{\varphi} Z_{\tilde{\varphi}} = 1$ and $Z_{\lambda} = 1$ to all orders as in (3.8).

Now let us turn our attention to the three-point function $\Gamma_{\tilde{\varphi}\varphi\varphi}$, which is given by the sum of all one-particle irreducible diagrams with two incoming legs and one outgoing leg. So $\Gamma_{\tilde{\varphi}\varphi\varphi} = k_{\parallel} I(\mathbf{k}_i, \omega_i)$, where $\mathbf{k}$ is the momentum of the external φ attaching to the outgoing leg. By power counting, $I(k_i, \omega_i) \sim \ln \Lambda + O(\Lambda^{-1})$ at $d = 5$, with a momentum-independent coefficient of $\ln \Lambda$ for each diagram. Such coefficients must then be the same for all choices of normalization point in evaluating the integrals, including the points where one of the incoming external momentum is zero. Conservation of momentum at such a vertex, together with the fact that the vertex is linearly dependent on momentum, leads to the cancellation of the coefficients of $\ln \Lambda$ between diagrams with different permutations of the outgoing arrow at that vertex. This we assume occurs order by order in perturbation theory. As a consequence, $Z_e = Z_{\parallel}^{-3/4}$ as in (3.8).

IV. Critical Exponent β

The exponent β is defined below T_c where the system undergoes phase separation. The nonlinear coupling $g\,\partial_{\perp}^2 \phi^3$ which would appear on the RHS of (2.2a) becomes important. Although, by power counting, g has a negative dimension above $d = 3$, it is a dangerous irrelevant variable which enters the calculation of the exponent β.

The dimension of g can be worked out of using (3.6): $g_0 \sim k_{\parallel}^{-1} k_{\perp}^{5-d} \sim r_{\parallel}^{1/2} k_{\perp}^{3-d}$, so that we define dimensionless parameters u_0 and u by

$$\begin{aligned} g_0 &= r_{0\parallel}^{1/2} \Lambda^{3-d} u_0 \\ u_0 &= \left(\frac{\kappa}{\Lambda}\right)^{3-d} Z_u u \end{aligned} \tag{A.15}$$

Argument similar to that in Appendix III shows that the four-point vertex function $\Gamma_{\tilde{\phi}\phi\phi\phi}$, with one insertion of the g-vertex, is UV convergent. Hence g_0 is not renormalized, which implies $Z_u = Z_{\parallel}^{-1/2}$. We then find

$$\begin{aligned}\beta_u(u_0, e_0) &= \left(\Lambda \frac{\partial}{\partial \Lambda} u_0\right)_{\text{ren}} \\ &= u_0[d-3+\tfrac{1}{2}\zeta_{\parallel}(e_0)]\end{aligned} \tag{A.16}$$

and at the nontrivial fixed point

$$\beta_u(u_0, e_0^*) = u_0 \tfrac{2}{3}(d-2) \tag{A.17}$$

to all orders in $\varepsilon = 5-d$. Thus the eigenvalue of u_0 at this fixed point is $-\frac{2}{3}(d-2)$.

Now we are ready to calculate β using the Callan–Symanzik equation for the magnetization G_ϕ, taking into account the effect of g_0. The C–S equation reads

$$\begin{aligned}&\left[\Lambda \frac{\partial}{\partial \Lambda} - \zeta_{\parallel}(e_0)\, r_{0_{\parallel}} \frac{\partial}{\partial r_{0_{\parallel}}} + \beta_e(e_0) \frac{\partial}{\partial e_0} + \beta_u(e_0, u_0) \frac{\partial}{\partial u_0}\right] \\ &\cdot G_\phi(r_{0_{\parallel}}, r_{0_{\perp}}, e_0, u_0, \Lambda) = 0\end{aligned} \tag{A.18}$$

At the nontrivial fixed point e_0^*, we obtain after dimensional analysis

$$\left[r_{0_{\perp}} \frac{\partial}{\partial r_{0_{\perp}}} - \left(1 - \frac{\varepsilon}{6}\right) - \frac{1}{2} \beta_u(e_0^*, u_0) \frac{\partial}{\partial u_0}\right] G_\phi = 0 \tag{A.19}$$

with solution

$$G_\phi(r_{0_{\parallel}}, r_{0_{\perp}}, e_0^*, u_0, \Lambda) = r_{0_{\perp}}^{1-\varepsilon/6} F(r_{0_{\perp}}^{1-\varepsilon/3} u_0) \tag{A.20}$$

where the dependence on $r_{0_{\parallel}}$, e_0^*, and Λ are suppressed in F. As we have scaled away from the critical region to get to this scaling form of G_ϕ, mean field theory can be applied to determine the asymptotic behavior of the scaling function $F(x)$ as $x \to 0$. As in the equilibrium case, we expect

$$F(x) \sim x^{-1/2} \qquad \text{as} \quad x \to 0$$

so $F(r_{0_{\perp}}^{1-\varepsilon/3} u_0)$ actually diverges as $u_0 \to 0$. u_0 is thus a dangerous irrelevant variable. We then identify the exponent β as $\frac{1}{2}$, true to all orders in ε for $d > 2$.

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